31. The Hagen-Poiseuille equation:

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31.1. Definition of variables and formulas

The Hagen–Poiseuille equation:

The Hagen–Poiseuille equation describes the flow of a Newtonian fluid (fluid with internal friction = viscosity) in laminar flow in a cylindrical pipe:

\[ I_v = -\frac{\pi}{8\eta} r^4 \frac{\Delta p}{\Delta l} \]

\( I_v \): volumetric flow rate \((m^3/s)\)
\( \eta \): viscosity \((Pa \cdot s)\)
\( r \): radius of the pipe \((m)\)
\( \Delta p / \Delta l \): pressure gradient, \( \Delta p \) pressure drop over \( \Delta l \) length

The volume of the fluid flowing through a cross section per unit time is directly proportional to the \( \Delta p / \Delta l \) pressure gradient and the fourth power of the pipe’s radius, and inversely proportional to the viscosity.

Similarly to Ohm’s law \((R=V/I)\), the resistance of a pipe can be defined: driving force / current (flow) = resistance

\[ R = \frac{\Delta p}{I_v} = -\frac{8\eta \Delta l}{\pi r^4} \]

The longer the pipe, the greater the viscosity and the smaller the radius the greater the resistance.

31.2. Sample exercises with solutions

31.2.1. Exercise

Assuming a constant pressure drop, to what fraction of the original value does the volumetric flow rate drop if the diameter of the blood vessel is reduced by 20% due to arteriosclerosis?

Since \( I_v \sim r^4 \), if \( r_2 = 0.8r_1 \), then \( I_{v2} = 0.8^4 \times I_{v1} = 0.4096 \times I_{v1} \), i.e. the volumetric flow rate drops to about 41% of the original value.
31.2.2. Exercise

The pulmonary artery, which connects the heart to the lungs, has an inner radius of 2.6 mm and is 8.4 cm long. If the speed of blood flow is 1.5 m/s, how much is the pressure drop between the heart and lungs in the artery? (The viscosity of blood is: \(2.7 \times 10^{-3} \text{ Pa}\times\text{s}\))

\[
I_V = -\frac{\pi r^4 \Delta p}{8\eta} \Rightarrow \Delta p = \frac{Av\eta\Delta l}{\pi r^4}
\]

\[
\frac{(2.6 \times 10^{-3})^2\pi \times 1.5 \times 8 \times 2.7 \times 10^{-3} \times 0.084}{\pi \times (2.6 \times 10^{-3})^4} = 403 \text{ Pa}
\]

31.2.3. Exercise

By how many \% was the diameter of the artery decreased by arteriosclerosis if the heart must maintain a blood pressure of 170 mmHg instead of the normal 120 mmHg to keep up the volumetric flow rate?

At constant volumetric flow rate and unchanged other parameters the following is true for the healthy and diseased artery:

\[
r_h^4\Delta p_h = r_d^4\Delta p_d = (\Delta p_h / \Delta p_d)^{0.25} = (120/170)^{0.25} = 0.917.
\]

Thus, a decrease of 8.3\% \((1-0.917)\) in the diameter (or radius) of the artery will require an increase of the pressure from 120 to 170 mmHg to maintain the volumetric flow rate.

31.2.4. Exercise

An infusion flask elevated above the point of injection injects 600 cm\(^3\) solution into the vein in 30 minutes through a needle of 0.25 mm radius and 5 cm length. The solution has a viscosity of \(1.2 \times 10^{-3} \text{ Pa}\times\text{s}\). Assume all the pressure drop occurs through the needle. What is the pressure generated by the height of the flask?

\[
I_V = -\frac{\pi r^4 \Delta p}{8\eta} \Rightarrow \Delta p = \frac{I_V\eta\Delta l}{\pi r^4} = \frac{\left(600 \times 10^{-6}\right) \times 8 \times 1.2 \times 10^{-3} \times 0.05}{\pi \times (2.5 \times 10^{-4})^4}
\]

\[
= 1.30 \times 10^4 \text{ Pa}
\]

31.2.5. Exercise

The pulmonary artery, which connects the heart to the lungs is 8.6 cm long. If the pressure drop between the heart and lungs in the artery is 410 Pa and the volumetric flow rate is \(3.2 \times 10^{-5} \text{ m}^3/\text{s}\)? (The viscosity of blood is: \(2.7 \times 10^{-3} \text{ Pa}\times\text{s}\)) What is the diameter of the artery?
\[ l_v = -\frac{\pi}{8\eta} r^4 \frac{\Delta p}{\Delta l} \implies \\
 d = 2 \times \left( \frac{l_v 8\eta \Delta l}{\pi \Delta p} \right)^{\frac{1}{4}} = 2 \times \left( \frac{3.2 \times 10^{-5} \times 8 \times 2.7 \times 10^{-3} \times 0.086}{\pi \times 410} \right)^{\frac{1}{4}} = 5.21 \text{ mm} \]

### 31.3. Practice exercises

1. Assuming a constant pressure drop, by how many percent does the volumetric flow rate decrease if the diameter of the blood vessel is reduced by 50% due to arteriosclerosis?
   (a) 25 %   (b) 50.45 %   (c) 75.50 %   (d) 93.75%

2. The pulmonary artery, which connects the heart to the lungs, has an inner radius of 2.6 mm and is 8.4 cm long. If the pressure drop between the heart and lungs is 400 Pa, what is the average speed of blood in the pulmonary artery? (The viscosity of blood is: \(2.7 \times 10^{-3} \text{ Pa} \cdot \text{s}\))
   (a) 0.8 m/s   (b) 1.5 m/s   (c) 2.2 m/s   (d) 3 m/s

3. The pulmonary artery, which connects the heart to the lungs, has an inner radius of 2.6 mm and is 8.4 cm long. If the pressure drop between the heart and lungs is 400 Pa, what is the volumetric flow rate of blood in the vessel? (The viscosity of blood is: \(2.7 \times 10^{-3} \text{ Pa} \cdot \text{s}\))
   (a) \(5 \times 10^4 \text{ m}^3/\text{s}\)   (b) \(8 \times 10^4 \text{ m}^3/\text{s}\)   (c) \(1.5 \times 10^5 \text{ m}^3/\text{s}\)   (d) \(3.2 \times 10^5 \text{ m}^3/\text{s}\)

4. What radius needle should be used to inject a volume of 500 cm\(^3\) of a solution into a patient in 30 min? Assume the length of the needle is 2.5 cm and the solution is elevated 1.0 m above the point of injection, which produces a pressure of \(9.8 \times 10^3 \text{ Pa}\) above the atmospheric pressure. Further, assume the viscosity (\(10^{-3} \text{ Pa} \cdot \text{s}\)) of the solution is the same as that of pure water, that the pressure inside the vein is atmospheric and all the pressure drop occurs through the needle.
   (a) 0.21 mm   (b) 0.42 mm   (c) 0.7 mm   (d) 1.2 mm

5. An infusion flask elevated 1 m above the point of injection produces \(9.8 \times 10^3 \text{ Pa}\) pressure and injects 500 cm\(^3\) solution into the vein in 30 minutes through a needle of 0.25 mm radius. The solution has the same density and viscosity (\(10^{-3} \text{ Pa} \cdot \text{s}\)) as water. Assume all the pressure drop occurs through the needle. What is the length of the needle?
   (a) 3.2 cm   (b) 3.8 cm   (c) 5.4 cm   (d) 6.2 cm

6. A hypodermic needle is 3.0 cm in length and 0.30 mm in diameter. What pressure difference between the input and output of the needle is required so that the flow rate of water through it will be 1 g/s? (Use \(1.0 \times 10^{-3} \text{ Pa} \cdot \text{s}\) as the viscosity of water.)
   (a) \(7 \times 10^4 \text{ Pa}\)   (b) \(1 \times 10^5 \text{ Pa}\)   (c) \(1.5 \times 10^5 \text{ Pa}\)   (d) \(2.2 \times 10^5 \text{ Pa}\)

7. A hypodermic needle is 3.0 cm in length and 0.30 mm in diameter. What is the mass of the water flowing through the needle in 1 min if the pressure difference between the input and output of the needle is \(1.2 \times 10^5 \text{ Pa}\)? (Use \(1.0 \times 10^{-3} \text{ Pa} \cdot \text{s}\) as the viscosity of water.)
   (a) 47.7 g   (b) 64.7 g   (c) 100.2 g   (d) 113.4 g
8. If the diameter of the blood vessel is reduced by 10% due to arteriosclerosis, then what blood pressure would be required instead of the healthy 120 mmHg if the heart attempted to maintain the volumetric flow rate?
(a) 142 mmHg  (b) 156 mmHg  (c) 183 mmHg  (d) 195 mmHg

9. How does the volumetric flow rate of blood change in an artery, if its diameter is halved and the pressure is increased by 50%?
(a) increases 4.52-fold  (b) does not change  (c) halved  (d) reduced to 9.4% of the original

10. How does the volumetric flow rate of a newtonian fluid change in a pipe, if its diameter is increased by 50% and its length is doubled?
(a) increases 4.52-fold  (b) increases 2.53-fold  (c) does not change  (d) reduced to 12.5% of the original

10. True or false?
During the flow of a Newtonian fluid, halving the radius of the tube while maintaining the pressure difference reduces the volumetric flow rate to 1/16th of the original value.

11. True or false?
During the flow of a Newtonian fluid, the volumetric flow rate is directly proportional to the fluid's viscosity.

12. True or false?
During the flow of a Newtonian fluid, the resistance of the tube is directly proportional to its length.

13. True or false?
During the flow of a Newtonian fluid, the resistance of the tube is inversely proportional to the fourth power of the radius.